

(On Automorphisms of Compact Complex Surfaces)

Copyright © 1986

(Ricardo Francisco Vila-Freyer)

TABLE OF CONTENTS

Chapter	Page
1. Introduction	1
2. Projective Space	2
3. Complex Tori	4
4. Minimal Rational Surfaces	6
5. Hopf Surfaces	10
6. Elliptic surfaces with no multiple singular fibres	15
7. Elliptic surfaces with multiple singular fibres	22
8. Inoue Surfaces	24
Bibliography	29

Acknowledgements

I want to thank Prof. S. Kobayashi for his valuable advice and guidance during the preparation of this work.

I am also thankful to all friends and professors whose support made this work possible. To the National University of Mexico for the support it gave me through the Institute of Mathematics and to the Department of Mathematics at the University of California at Berkeley.

I dedicate this work to the memory of my father. To my mother, sisters and brother.

1. Introduction

The object of this work is to study automorphism groups of compact complex surfaces.

For a compact complex manifold, in general, this group is a complex Lie group, and its Lie algebra consists of the holomorphic vector fields. If the manifold M is of general type, then the group $\text{Aut}(M)$ is finite. (c.f. Kobayashi [10], and Kobayashi and Ochiai [11]). It follows from a result of Bandman [2] that if M has negative first Chern class, (i.e. ample canonical bundle), then $\text{Aut}(M)$ is bounded by a bound that depends only on $(-1)^n c_1(M)$ and $K_M^{m_X}$ where $n = \dim M$, and m_X is the least integer m such that K_X^m is very ample. But he does not give any estimate for this bound.

In a relatively old paper Andreotti [1] gives a bound of $\text{Aut}(M)$ when M is a complex surface with negative first Chern class. More recently, Howard and Sommese [8] use his proof for the general case, when M is a compact complex manifold with $c_1(M) < 0$. However, applied to curves, their estimate is much weaker than the classical result of Hurwitz.

Fujiki [6] proves that if M is the image of a compact Kaehler manifold, then there is an exact sequence of complex Lie groups

$$0 \rightarrow L \rightarrow \text{Aut}(M) \rightarrow T \rightarrow 0$$

where L is meromorphically equivalent to a linear group, T a complex torus. This gives the best description of the structure of the group of automorphisms of M .

This work consists of describing the structure of automorphism groups of compact surfaces. Essentially I give a description for Hopf surfaces (which are non Kaehler, and of algebraic dimension one or zero), for elliptic surfaces, with the exception of elliptic K3, and elliptic complex tori; and for the Inoue surfaces of Kodaira class VII₀. I include a description for projective space, complex tori and the Hirzebruch rational surfaces, although this is already well known, because the structure of the group of automorphisms is similar, or we need to make use of it.

For K3 surfaces the problem is still not solved in general, but much has been done by Nikulin [14]. And using his results, Barth and Peters [3] have worked on the automorphisms of Enriques surfaces.

2. Projective Space

(c.f. Griffiths and Harris [7])

1. Notice first that any submanifold V of \mathbf{P}^n which is homologous to a hyperplane H is indeed a hyperplane: If V is homologous to H and L is any line in \mathbf{P}^n , then the intersection number $i(V,L)=1$. If p and q are in V then the line L passing through p and q gives also $i(V,L)\geq 2$, unless L is contained in V . Since $i(V,L)=1$, we have that L is contained in V . Hence V is a linear subspace of \mathbf{P}^n .

2. Any biholomorphic automorphism is induced by a linear transformation of \mathbf{C}^{n+1} . If (y_0, \dots, y_n) are homogeneous coordinates of \mathbf{P}^n , write $x_i = y_i / y_0$ in $\mathbf{P}^n - H$ where $H = \{y_0 = 0\}$. We know that the fundamental class of a hyperplane in \mathbf{P}^n generates $H^2(\mathbf{P}^n, \mathbf{Z})$. So if f is in $\text{Aut}(\mathbf{P}^n)$, $f(H)$ is homologous to a hyperplane for any given hyperplane H , and $f(H)$ is a hyperplane by 1. Composing f with a linear transformation we may assume that $f(H) = H$. If $H_i = \{y_i = 0\}$ are hyperplanes then

$$f(H_i) = \{a_{1i}x_1 + \dots + a_{ni}x_n = 0\}$$

and the pull back of x_i , $f^*(x_i)$ is a meromorphic function on \mathbf{P}^n with a single pole along H and a single zero along $f(H_i)$ hence $\frac{f^*(y_i)}{a_{0i} + a_{1i}x_1 + \dots + a_{ni}x_n}$ is holomorphic on all of \mathbf{P}^n and then constant. Therefore f is linear.

3. So we conclude that any biholomorphic automorphism of \mathbf{P}^n is induced by a linear transformation of \mathbf{C}^{n+1} . Obviously the transformations T and λT induce the same transformation on \mathbf{P}^n . Hence:

2.1. Theorem:

The group of biholomorphic transformations of the projective space of n -dimension is given by $PGL(n, \mathbf{C}) = GL(n+1, \mathbf{C}) / \langle \mu I \rangle$, where μ is any non-zero complex number, and I is the identity matrix.

3. Complex Tori

If Λ is a lattice of rank $2n$ in \mathbb{C}^n , let $T^n = \mathbb{C}^n / \Lambda$ denote the complex torus of dimension n with period Λ . T itself is a complex Lie group, and the connected component $\text{Aut}_0(T)$ of the group of automorphisms of T is the same group T , acting by translation. To find $\text{Aut}(T)$ we need to describe the isotropy group at one element, say the identity e of T . For this, note that any map $X: T \rightarrow T$ such that $X(e) = e$ is induced by a linear map on \mathbb{C}^n :

We denote by $\Pi = (\lambda_1, \dots, \lambda_{2n})$ the matrix whose columns are a basis for the lattice Λ , and z_1, \dots, z_n , the coordinates of T^n induced from \mathbb{C}^n . We denote the matrix X in terms of the basis $\{z_i\}$.

The map X induces a linear map $M: H_1(T^n, \mathbb{Z}) \rightarrow H_1(T^n, \mathbb{Z})$. But this homology group is isomorphic to the lattice Λ . Since M is a \mathbb{Z} -module isomorphism, it has integer coefficients, and X and M act both on Π . So we have:

$$X\Pi = \Pi M$$

(We may think of it in this terms: the differential structure of T^n is $\mathbb{R}^{2n} / \mathbb{Z}^{2n}$, and the matrix Π is the matrix of change of basis from the standard basis for the lattice \mathbb{Z}^{2n} to the lattice Λ . So that we have the map $\Pi: \mathbb{R}^{2n} / \mathbb{Z}^{2n} \rightarrow T^n = \mathbb{C}^n / \Lambda$ and the map X acts on T^n , and M on $\mathbb{R}^{2n} / \mathbb{Z}^{2n}$. And they act preserving the map Π).

Hence $X \in SL(n, \mathbb{C})$ and $M \in SL(2n, \mathbb{Z})$.

We can normalize Π in such a way that that it is equivalent to (I, Ω) , where I is the identity matrix and Ω is a complex $(n \times n)$ -matrix such that $\det(\text{Im } \Omega) \neq 0$. Then using the same notation: If $X(e) = e$ then there exists a matrix $M \in SL(2n, \mathbb{Z})$ such that

$$X(I, \Omega) = (I, \Omega)M$$

so if we write

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in SL(2n, \mathbb{Z})$$

where A, B, C and D are each integral $(n \times n)$ -matrices. We obtain

3.1. Theorem:

For a complex torus $T = \mathbb{C}^n / \Pi$ with $\Pi = (I, \Omega)$, the isotropy group at the identity $e \in T^n$ is isomorphic to the solutions of the integral matrix equation

$$\Omega = (A + \Omega B)^{-1}(B + \Omega C),$$

where

$$M = \begin{bmatrix} AB \\ CD \end{bmatrix} \in SL(2n, \mathbb{Z}).$$

We always have the solutions $M = \pm I_{2n}$. In the case of curves it is known that this has other solutions if and only if $\Omega = i$ or $\Omega = e^{2\pi i/3}$, in which case M is in a subgroup of order 4 or 6 respectively.

For higher dimensions, I do not know how to solve this equation, (in the case of abelian varieties this is related to the concept of Complex Multiplication). Nevertheless it is trivial to check that if we denote T_i and T_p , the elliptic curves whose period matrices are $(1, i)$ and $(1, e^{2\pi i/3})$ respectively, then their products: $T_i \times T_i$, $T_i \times T_p$, and $T_p \times T_p$, have non trivial isotropy groups at e of orders greater than two.

4. Minimal Rational Surfaces

It is known that the minimal surfaces birational to \mathbf{P}^2 are \mathbf{P}^2 itself and the Hirzebruch rational surfaces which we proceed to describe.

If we denote by $H^{-1} = \{(\lambda, x) \in \mathbf{P}^1 \times \mathbf{C}^2 : x \in \lambda\}$ the tautological line bundle over \mathbf{P}^1 , H its dual line bundle called the hyperplane line bundle and by \mathbf{C} the trivial line bundle, then the Hirzebruch surface F_n is the projectivization of the vector bundle: $L = H^n + \mathbf{C}$. That is,

$$\pi: F_n = P(H^n + \mathbf{C}) \rightarrow \mathbf{P}^1$$

For each n , F_n is characterized by its special divisors: E_0 is the divisor induced by the section $(0,1)$ of $L = H^n + \mathbf{C}$. E_σ is induced by the section $(\sigma, 1)$ of $L = H^n + \mathbf{C}$, where σ is an element of $H^0(\mathbf{P}^1, \mathcal{O}(H^n))$. E_∞ is induced by $\{(x, 0) : x \in H^n\}$ and, of course, C_z is the fibre $\pi^{-1}(z)$ with z an element of \mathbf{P}^1 .

Then the following is known: E_0 is homologous to E_σ and we have the following intersection properties: $E_0 E_0 = n$, $E_\infty E_\infty = -n$, $C_z C_w = 0$ for z, w two points of \mathbf{P}^1 , and $C_z E_0 = 1$.

The homology classes of E_0 and C_z are the generators of $H^2(F_n, \mathbf{Z})$. In particular, we have that $E_\infty = E_0 - nC_z$. Also F_n has only one irreducible rational curve of self intersection $-n$. This differentiates F_n from F_m when n and m are different.

Notice that the intersection properties characterize the fibration on F_n in a unique way, that is, the fibres of $F_n \rightarrow \mathbf{P}^1$ are the only possible fibration of F_n . In fact, if there exists a different fibration of F_n , and if we denote by D one of its fibres, then $D = aE_0 + bC_z$ and we have

$$0 < DE_\infty = (aE_0 + bC_z)(E_0 - nC_z) = b$$

$$0 < DC_z = (aE_0 + bC_z)C = a$$

and

$$0 = DD = (aE_0 + bC)^2 = a^2n + 2ab$$

but $a > 0$, $b > 0$ and $n > 0$ make the latter impossible. So D must be homologous to a

fibre of $\pi: F_n \rightarrow \mathbf{P}^1$. Hence D is a fibre. This implies:

4.1. Theorem:

(a) If $n=0$, $F_0 = \mathbf{P}^1 \times \mathbf{P}^1$ and $Aut(\mathbf{P}^1 \times \mathbf{P}^1) = PGL(1) \times PGL(1) \times \mathbf{Z}_2$.

(b) For $n \geq 1$, the following exact sequence holds:

$$0 \rightarrow T \rightarrow Aut(F_n) \rightarrow PGL(1) \rightarrow 0,$$

where T is given by $T = H^0(\mathbf{P}^1, \mathcal{O}(H)) \rtimes \mathbf{C}^*$, with \rtimes denoting the semidirect product and \mathbf{C}^* acting by multiplication on the first factor.

Proof:

(a) If $n=0$, then $F_0 = \mathbf{P}^1 \times \mathbf{P}^1$ since H^0 is the trivial line bundle, and $H_2(\mathbf{P}^1 \times \mathbf{P}^1, \mathbf{Z}) = \mathbf{Z} + \mathbf{Z}$ is generated by $a = \mathbf{P}^1 \times \{pt\}$ and $b = \{pt\} \times \mathbf{P}^1$. Now, if $f \in Aut(\mathbf{P}^1 \times \mathbf{P}^1)$, then

$$f \cdot a = \alpha a + \beta b$$

$$f \cdot b = \gamma a + \delta b$$

where $M = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ is an integral matrix with determinant ± 1 , because it is the matrix of the induced map f_* on the integral homology, with respect to the basis given by a and b . Notice that $aa = bb = 0$ and $ab = 1$, so we have that

$$0 = f_*(a) f_*(a) = (\alpha a + \beta b)^2 = 2\alpha\beta,$$

$$0 = f_*(b) f_*(b) = (\gamma a + \delta b)^2 = 2\gamma\delta,$$

and

$$\pm 1 = f_*(a) f_*(b) = \alpha\delta - \gamma\beta = \det M.$$

Then necessarily:

i) $\beta=0$ and $\gamma=0$

or

ii) $\alpha=0$ and $\delta=0$.

giving $\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$, or $\begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix}$.

We may compose f with $i(z,w) = (w,z)$ to obtain an automorphism inducing a map

as in i) in homology. Then we have that $bf_*(b)=0$, with $b=\{pt\}\times\mathbf{P}^1$. If we take a point $(x,y)\in f(b)$ then $(x,y)\in f(b)\cap\{x\}\times\mathbf{P}^1$, but this means that $f(b)$ is contained in $\{x\}\times\mathbf{P}^1$, i. e. $f(\{pt\}\times\mathbf{P}^1)=\{pt\}\times\mathbf{P}^1$, and we get that f is in $PGL(1,\mathbf{C})\times PGL(1,\mathbf{C})$. Hence, an automorphism inducing a map as in ii) on homology, corresponds to $i(z,w)=(w,z)$ composed with a map of $PGL(1)\times PGL(1)$. And we have that the automorphism $i(z,w)=(w,z)$, together with $PGL(1)\times PGL(1)$ form all the automorphisms of $\mathbf{P}^1\times\mathbf{P}^1$.

- (b) In this case, the \mathbf{P}^1 fibration $F_n \rightarrow \mathbf{P}^1$ is unique. Hence any automorphism of F_n maps fibres into fibres. So it induces an automorphism of \mathbf{P}^1 , and all automorphisms of \mathbf{P}^1 can be obtained in this way. Any automorphism of \mathbf{P}^1 induces an automorphism of $\mathbf{C}^2-\{0\}$ and also induces an automorphism of the line bundle H^{-1} , with it we can get an automorphism of any line bundle of \mathbf{P}^1 which induces the original automorphism on the base and commutes with the projection map of the line bundle. This induces a map of F_n with the same properties. So we have that $Aut(F_n) \rightarrow PGL(1,\mathbf{C})$ is onto.

Now if T is the kernel, we want to prove: $T=H^0(\mathbf{P}^1, O(H))\rtimes\mathbf{C}^*$, with \rtimes denoting the semidirect product and \mathbf{C}^* acting by multiplication on the first factor. Let C be a fibre and E_0 be the zero section as described above. They form a basis of $H_2(F_n, \mathbf{Z})$. Notice that the uniqueness of the divisor E_∞ forces it to be mapped into itself by any automorphism f of F_n , and if $f(C_z)=C_z$ holds for all $z\in\mathbf{P}^1$, then f acts as the identity on E_∞ . Let $f\in Aut(F_n)$, and $f(E_0)=E_f$. We know that E_f is homologous to aE_0+bC . Then using the intersection properties of the generators we get: $E_f E_f = n$ and $E_f C = 1$ (since f maps fibres into fibres). so

$$n = E_f E_f = aE_0 E_f + bCE_f = aE_0 E_f + b.$$

On the other hand, we have

$$1 = E_f C = aE_0 C + bCC = aE_0 C = a$$

and

$$E_f E_0 = E_0 E_0 + b C E_0 = n + b$$

obtaining $b=0$, and E_f is homologous to E_0 .

Suppose that f sends each fibre into itself i.e. $f \in T$. Notice also that since f maps each fibre into itself, leaving the "section at infinity" fixed, then $P(H^n + C) - E_\infty = L$ is just a line bundle, and f consists of automorphisms of this line bundle acting on each fibre. Also the zero section E of this line bundle L has self intersection n . So that L is equivalent to H^n as line bundle over P^1 . In such a way that $P(H^n + C)$ is really the completion of the fibres of the line bundle H^n . So, we have that for any x in L_z , ($z \in P^1$) f acts as $x \rightarrow a(z)x + b(z)$ with a, b in $H^0(P^1, O(H^n))$, and, since a can never vanish, $a(z)$ is constant; i.e. we have that f corresponds to

$$(b(z), a) \in H^0(P^1, O(H^n)) \times C^*$$

and the composition law will be if f_i corresponds to $(b_i(z), a_i)$, then

$$f_2 f_1(x) = f_2(a_1 x + b_1(z)) = a_2 a_1 x + a_2 b_1(z) + b_2(z) = f(x)$$

where f corresponds to

$$(a_2 b_1(z) + b_2(z), a_2 a_1).$$

Obtaining that T is isomorphic to $H^0(P^1, O(H^n)) \times C^*$. Q.E.D.

For notations and properties of the Hirzebruch rational surfaces I followed Griffiths and Harris [7], but the description of their automorphism group is a problem in Beauville [4].

5. Hopf Surfaces

A Hopf surface is defined to be a compact complex surface which has $W = \mathbb{C}^2 - \{0\}$ as its universal covering. Then all Hopf surfaces are biholomorphic to W/G , where G is some group of biholomorphic automorphisms of \mathbb{C}^2 leaving 0 fixed. The main reference in this section is Kodaira [13]. We quote some properties of these surfaces:

1. There exists an element $g \in G$ which is a contraction of \mathbb{C}^2 , such that g^n belongs to the center of G , for some integer n . Any Hopf surface S whose fundamental group G is generated by one element g is called a primary Hopf surface. Then there exists a contraction f such that $S = W/\langle f \rangle$. (Where $\langle f \rangle$ denotes the group generated by f).

2. Any Hopf surface belongs to Kodaira class VII₀. This is the class of minimal compact complex surfaces with $b_1=1$, $q=1$ and $p_g=0$. And, it has a finite unramified normal covering surface which is a primary Hopf surface.

3. Any primary Hopf surface S is a quotient $W/\langle f \rangle$ and there exists a system of global complex coordinates on \mathbb{C}^2 such that f has the form:

$$f(z_1, z_2) = (\alpha_1 z_1 + \lambda z_2^m, \alpha_2 z_2)$$

with m a positive integer, and the constants α_1, α_2 , and λ satisfy:

$$(\alpha_1 - \alpha_2^m)\lambda = 0 \quad (*)$$

and

$$0 < |\alpha_1| \leq |\alpha_2| < 1. \quad (**)$$

Remark: A compact complex surface S is called elliptic if there exists a holomorphic map $f: S \rightarrow C$ where C is a compact complex curve, and all but a finite number of fibres of f are elliptic curves.

4. A Hopf surface $S = W/\langle f \rangle$ is elliptic if and only if $\lambda = 0$ and there exist positive integers k and l such that $\alpha_1^k = \alpha_2^l$. We are assuming that f is as in (3).

5. If a Hopf surface $S = W/G$ is elliptic and G is non abelian then there are global coordinates on \mathbb{C}^2 such that G is a subgroup of $GL(2, \mathbb{C})$.

6. If a Hopf surface $S=W/G$ is not elliptic, then G is a direct product of an infinite cyclic group Z and a finite cyclic group Z_d of order d , $G=Z+Z_d$. Z is generated by a contraction $f(z_1, z_2)=(\alpha_1 z_1 + \lambda z_2^m, \alpha_2 z_2)$ with conditions on the constants as in (*) and (**) of (3). Z_d is generated by a linear transformation $c(z, w)=(\epsilon_1 z, \epsilon_2 w)$ where $(\epsilon_1 - \epsilon_2^m)\lambda = 0$ and ϵ_1, ϵ_2 are primitive d -roots of unity.

7. Notice also that if ϕ is any automorphism of a Hopf surface $S=W/G$, then ϕ lifts to an automorphism $\tilde{\phi}$ of \mathbb{C}^2 leaving the origin fixed, and such that for any $g \in G$ there is some $g' \in G$ such that $\tilde{\phi}g = g'\tilde{\phi}$. This is just a consequence of the lifting property in topology.

8. Finally notice that if $f(z_1, z_2)=(\alpha_1 z_1 + \lambda z_2^m, \alpha_2 z_2)$ and $\lambda \neq 0$, then equation (*) implies

$$f^r(z_1, z_2) = (\alpha_1^r z_1 + r \alpha_1^{r-1} \lambda z_2^m, \alpha_2^r z_2)$$

So we consider now the two possible cases according as the algebraic dimension of the Hopf surface S is one (S is elliptic) or zero (S has no non-trivial meromorphic functions)

I. Suppose that $S=W/G$ is non elliptic. We know that $G=Z+Z_d$ with Z generated by f and Z_d by g as in 6. Assume $\phi \in \text{Aut}(S)$. Then ϕ lifts to an automorphism of \mathbb{C}^2 which we will denote by the same letter.

a) Assume also that $\lambda \neq 0$. Suppose the Taylor expansion of ϕ is given by

$$\phi(z_1, z_2) = (\sum a_{i_1, i_2} z_1^{i_1} z_2^{i_2}, \sum b_{j_1, j_2} z_1^{j_1} z_2^{j_2}).$$

Observe that

$$\phi g = g^r \phi$$

for some integer r .

Otherwise there would be integers r, k such that

$$\phi g = f^r g^k \phi,$$

but then

$$\phi g^{dn} = f^{rdn} g^{kdn} \phi,$$

i.e. $\phi = f^{rdn} \phi$, and setting $n \rightarrow \infty$ we get that ϕ is constant, unless $r=0$, which is a contradiction. (recall that f is a contraction). So $\phi g = g^k \phi$. Similarly, $\phi f^d = f^{rd} \phi$ (d is the order of G). Suppose $f^d = (az_1 + \mu z_2^m, bz_2)$ and $\mu \neq 0$. Then we have as before that $a = b^m$ and we get $f^{rd} = (a^r z_1 + r a^{r-1} \mu z_2^m, b^r z_2)$. So that

$$\phi f^d(z_1, z_2) = (\sum a_{i_1, i_2} (az_1 + \mu z_2^m)^{i_1} (bz_2)^{i_2}, \sum b_{j_1, j_2} (az_1 + \mu z_2^m)^{j_1} (bz_2)^{j_2})$$

and

$$f^{rd} \phi(z_1, z_2) = (a^r \sum a_{i_1, i_2} z_1^{i_1} z_2^{i_2} + r a^{r-1} \mu (\sum b_{j_1, j_2} z_1^{j_1} z_2^{j_2})^m, b^r \sum b_{j_1, j_2} z_1^{j_1} z_2^{j_2}).$$

Notice: On the first entry the degree of a is r and $r-1$ in a second term and the degree of μ is one. On the second entry there is no a or μ and the degree of b is r . So: $b_{i_1, i_2} = 0$ unless $j_1 = 0$ and $j_2 = r$. From the first entry we can conclude $r=1$, i.e. ϕ and f^d commute; since otherwise there would be terms on a^j with $0 \leq j \leq r$. So we get that $b_{01} = \text{const.}$, and $b_{j_1, j_2} = 0$ for all other j_1, j_2 . Rewriting we get:

$$\phi f^d(z_1, z_2) = (\sum a_{i_1, i_2} (az_1 + \mu z_2^m)^{i_1} (bz_2)^{i_2}, b_{01} bz_2)$$

$$f^d \phi(z_1, z_2) = (a \sum a_{i_1, i_2} z_1^{i_1} z_2^{i_2} + \mu (b_{01} z_2)^m, b b_{01} z_2)$$

this means that $a_{i_1, i_2} = 0$ unless: either $i_1 = 1$ and $i_2 = 0$ then $a_{10} = b_{01}^m$; or $i_1 = 0$ and $i_2 = m$ then a_{0m} is arbitrary. Since $b^m = a$, we get:

$$\phi(z_1, z_2) = (Az_1 + \mu z_2^m, Bz_2) \quad (***)$$

with $B^m = A$, i.e. $\text{Aut}(S) = \{ \phi : \phi \text{ is as } (***) \}$

b) Assume that $\lambda = 0$. Since S is non elliptic, we have that $\alpha_1^q \neq \alpha_2^p$ for all p, q non-zero integers, and

$$\begin{aligned} \phi f^d(z_1, z_2) &= (\sum a_{i_1, i_2} (az_1)^{i_1} (bz_2)^{i_2}, \sum b_{j_1, j_2} (az_1)^{j_1} (bz_2)^{j_2}) = \\ &= f^{rd} \phi(z_1, z_2) = (a^r \sum a_{i_1, i_2} z_1^{i_1} z_2^{i_2}, b^r \sum b_{j_1, j_2} z_1^{j_1} z_2^{j_2}) \end{aligned}$$

So $r=1$, and ϕ must actually be of the form $\phi(z_1, z_2) = (Az_1, Bz_2)$ so that

$$\text{Aut}(S) = \{ \phi : \phi(z, w) = (Az, Bw) \}.$$

As conclusion we obtain:

5.1. Theorem:

If $S=W/G$ is a non-elliptic Hopf surface, where the group G is described as in 6), then

i) $\text{Aut}(S)=\{\phi : \phi(z, w)=(Az + \mu w^m, Bw) \text{ with } B^m=A \}$ if $\lambda \neq 0$,

and

ii) $\text{Aut}(S)=\{\phi : \phi(z, w)=(Az, Bw) \}$ if $\lambda=0$.

(In both cases A, B are complex numbers).

II. We assume now that $S=W/G$ is an elliptic Hopf surface. Then its fundamental group G may be abelian or not.

a) Assume that G is abelian. then $G=Z+Z_{l_1}+\dots+Z_{l_k}$ and the cyclic group Z is generated by $f(z, w)=(\alpha_1 z, \alpha_2 w)$ so that $\lambda=0$ and there are positive integers p and q such that $\alpha_1^p=\alpha_2^q$. Again as before, let $\phi \in \text{Aut}(S)$ then $\phi f^m = f^{nr} \phi$ where $m = \text{m.c.m.}\{l_1, \dots, l_k\}$.

First check that $r=1$:

we write as before,

$$\phi(z_1, z_2) = (\sum a_{i_1, i_2} z_1^{i_1} z_2^{i_2}, \sum b_{j_1, j_2} z_1^{j_1} z_2^{j_2})$$

then

$$\phi f^m(z_1, z_2) = (\sum a_{i_1, i_2} \alpha_1^{mi_1} \alpha_2^{mi_2} z_1^{i_1} z_2^{i_2}, \sum b_{j_1, j_2} \alpha_1^{mj_1} \alpha_2^{mj_2} z_1^{j_1} z_2^{j_2}).$$

$$\phi^{rm} f(z_1, z_2) = (\alpha_1^{rm} \sum a_{i_1, i_2} z_1^{i_1} z_2^{i_2}, \alpha_2^{rm} \sum b_{j_1, j_2} z_1^{j_1} z_2^{j_2})$$

If we suppose $\alpha_1^p=\alpha_2^q$ and p and q are the smallest positive integers with this property,

then

$$\alpha_1^{mi_1} \alpha_2^{mi_2} = \alpha_1^{rm}$$

hence

$$\alpha_2^{i_2} = \alpha_1^{r-i_1}$$

and in a similar way $\alpha_1^{mj_1} \alpha_2^{mj_2} = \alpha_2^{rm}$, giving that $\alpha_{j_1} = \alpha_2^{r-j_2}$, consequently q divides i_2 , p divides $r-i_1$ and so on. i.e. $i_2=nq$, $r-i_1=mp$ and similarly $j_1=\tilde{n}p$ and $r-j_2=\tilde{m}p$ so that

$$\phi(z_1, z_2) = \left(\sum_{n: r-nq \geq 0} a_n z_1^{r-nq} z_2^{np}, \sum_{n: r-np \geq 0} b_n z_1^{nq} z_2^{r-np} \right)$$

and also $\phi f = f \circ \phi$, but checking for the point (1,0) we see that $r=1$, and necessarily $n=0$ in the series expansion of ϕ whenever p or q are different from 1. Hence $\text{Aut}(S) = \{f : f(z, w) = (Az, Bw)\}$.

Or $n=0$ and $n=1$ are the only two choices in the power series expansion if $p=q=1$; giving that ϕ is linear and $\text{Aut}(S) = \{f \in GL(2, \mathbb{C})\}$.

b) Now assume that $S=W/G$ is elliptic and G is non abelian. Then Kodaira [13] proves that there exist global coordinates such that G is a subgroup of $GL(2, \mathbb{C})$. So, it is easy to see that the normalizer of G in $GL(2, \mathbb{C})$, denoted N_G , is a group of biholomorphic automorphisms of S . We want to verify that this is the only possibility, i.e. if $\phi \in \text{Aut}(S)$ then ϕ lifts to a linear transformation of \mathbb{C}^2 .

But again we know that there exists a primitive normal covering of S , $W/\langle f \rangle$ where $f(z, w) = (\alpha_1 z, \alpha_2 w)$ with $\alpha_1^p = \alpha_2^q$ so we have seen that if $\phi \in \text{Aut}(S)$, then it induces an automorphism of \mathbb{C}^2 , then it induces an automorphism of $W/\langle f \rangle$; so that ϕ is in the normalizer of $\langle f \rangle$ in $GL(2, \mathbb{C})$ consequently ϕ is linear. So we obtain:

5.2. Theorem:

Assume that $S=W/G$ is an elliptic Hopf surface. Then the group of biholomorphic automorphisms is:

a) $\text{Aut}(S) = \{f \in GL(2, \mathbb{C}) : f \text{ commutes with the elements of } G\}$, if G is abelian;

and

b) $\text{Aut}(S)$ is the normalizer of G in $GL(2, \mathbb{C})$, if G is not abelian.

6. Elliptic surfaces with no multiple fibres

To describe their basic properties, we follow Kodaira [11] closely. Assume that $\pi: S \rightarrow \Delta$ is an elliptic surface with no multiple singular fibres and no exceptional curve on a fibre. Since most of the fibres of π except for a finite number of them are elliptic curves, we can define a meromorphic function on Δ denoted by J in the following way:

If $z \in \Delta$ is a regular point, we can write

$$\pi^{-1}(z) = \mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z}) \quad \text{with } \text{Im}(\tau) > 0.$$

We denote by J the elliptic modular function. Then by abuse of notation $J(z)$ is defined to be $J(\tau)$, (it does not depend on the value of τ defining the elliptic curve $\pi^{-1}(z)$). So far the definition makes sense only for regular values of π . Now the function J can be extended to a meromorphic function on Δ . This is called the functional invariant of the elliptic fibration. In a similar way, define a sheaf G over Δ by defining the stalks over the regular points as $G_z = H_1(\pi^{-1}(z), \mathbf{Z})$, and extending this sheaf in an appropriate way to all of Δ . This is called the group invariant of the fibration. Then we consider the family $F(J, G)$ of elliptic surfaces which have J and G as their functional and group invariants respectively.

In each family $F(J, G)$ there exists a unique surface B (up to a biholomorphic equivalence) such that $\psi: B \rightarrow \Delta$ has a section $\sigma: \Delta \rightarrow B$. Then any $S \in F(J, G)$ is obtained from B in the following way: Denote by $B^\#$ the non-compact surface obtained from B by deleting: 1) The points of B for which the map ψ is not regular and 2) the components of the singular fibres that have multiplicity greater than one. Then $\psi^{-1}(z) \cap B^\#$ is either a regular fibre (if z is a regular value of ψ) or the union of several copies of either \mathbf{C} or $\mathbf{C} - \{0\}$. Now $B^\# \rightarrow \Delta$ has the structure of an analytic family of abelian Lie groups over Δ such that the identity element on each fibre $\psi^{-1}(z)$ is precisely the value of the section $\sigma(z)$.

Let σ be an analytic section of $B^\#$ over some open subset U of Δ . Then the automorphism of $\psi^{-1}(z) \cap B^\#$ defined by $z \rightarrow z + \sigma(\psi(z))$ has a unique extension to an auto-

morphism of $\psi^{-1}(z)$. Denote by $\Omega(B^\#)$ the sheaf of germs of sections of $B^\#$ over Δ , and $\sigma \in H^1(\Delta, \Omega(B^\#))$. Then we obtain an elliptic surface B^σ as follows: Choose a 1-cocycle $\{\sigma_{ij}\}$ representing σ , and an open cover $\{U_i\}$ of Δ for this cocycle. Then

$$B^\sigma = \bigcup \psi^{-1}(U_i) / \approx,$$

where the equivalence relation \approx is that $z_i \in \psi^{-1}(U_i)$ and $z_j \in \psi^{-1}(U_j)$ are equivalent if and only if $z_i = L_{ij}(z_j)$, and $L_{ij}: \psi^{-1}(U_{ij}) \rightarrow \psi^{-1}(U_{ij})$ is the unique extension of the automorphism $z \rightarrow z + \sigma_{ij}(\psi(z))$. Notice that the fibres of B^σ are the same as those of B ; but they are "glued" in a twisted way. Then the important result is that any elliptic surface S in $F(J, G)$ is obtained from B as $S = B^\sigma$ for some $\sigma \in H^1(\Delta, \Omega(B^\#))$.

As for the invariants of B , we denote by L the normal line bundle of the section $\sigma(\Delta)$ in B , and by K the canonical line bundle of Δ . Then $c_1(L) = -(p_g + 1)$, ([11] Thm. 12.3); and the Euler number of B is equal to the sum of the Euler numbers of the singular fibres. Hence it is zero if and only if there are no singular fibres.

Let S be an elliptic surface (with no restriction on the fibres) and denote by $\{a_\rho\}$ the subset of points on Δ which have singular fibres, and by $\Delta' = \Delta - \{a_\rho\}$. Any two non-singular fibres are homotopic. Denote by $C_z = \psi^{-1}(z)$ the fibre over $z \in \Delta$. It is also true that C_z with $z \in \Delta'$ is homologous to C_{a_ρ} (c.f. [12] sect. 4), in fact the divisor a_ρ is linearly equivalent to a divisor of the form $\sum n_i z_i$ on Δ , with $z_i \in \Delta'$ and $\sum n_i = 1$. Therefore the divisor C_{a_ρ} is linearly equivalent to $\sum n_i C_{z_i}$, while each C_{z_i} is homotopic to C_z for some $z \in \Delta'$. So we get that C_{a_ρ} and C_z are homologous.

Suppose that S is an elliptic surface with no multiple singular fibres, $S = B^\sigma$, where $\sigma \in H^0(\Delta, \Omega(B^\#))$ and $\pi: S \rightarrow \Delta$ is the elliptic fibration, $\psi: B \rightarrow \Delta$ is the basic member with a section $\sigma: \Delta \rightarrow B$ of the family $F(J, G)$ containing S .

6.1. Lemma:

If G is the maximal subgroup of $\text{Aut}(S)$ which acts by automorphisms on each fibre (i.e. it induces the identity map on the base curve Δ), then G is isomorphic to $H^0(\Delta, \Omega(B^\#))$.

Proof:

Let $f : B \rightarrow B$ be such that $f(C_z) = C_z$ for all fibres C_z of B . Then f induces a section $\tau \in H^0(\Delta, \Omega(B^\#))$ by setting

$$\tau(z) = f(\sigma(z)) \quad \text{for } z \in \Delta,$$

and conversely, any section τ induces an automorphism f acting on each fibre C_z as follows:

$$f(w) = w + \tau(\psi(w)) \quad \text{for all } w \in C_z.$$

I will denote by τ the section related to the automorphism f .

Suppose now that $S = B^\sigma$. We want to verify that the group of automorphisms acting on the fibres of S is the same as the one acting on the fibres of B .

First assume f is as above. Then, if $\{U_i\}$ is the covering of Δ associated to the cocycle $\{\sigma_{ij}\}$ representing σ , such that $S = B^\sigma = \bigcup \psi^{-1}(U_i) / \sim$ with z_i equivalent to z_j if and only if $z_i = L_{ij}(z_j) = z_j + \sigma_{ij}(\psi(z_j))$, then we have that $f(z_i)$ is equivalent to $f(z_j)$ because:

$$\begin{aligned} f(L_{ij}(z_j)) &= L_{ij}(z_j) + \tau(\psi(L_{ij}(z_j))) \\ &= z_j + \tau(\psi(z_j)) + \sigma_{ij}(\psi(z_j)) \\ &= f(z_j) + \sigma_{ij}(\psi(f(z_j))) = L_{ij}(f(z_j)). \end{aligned}$$

Conversely, if f is an automorphism of S acting on the fibres, then from the description of S as B^σ , $z_i \approx z_j$ if and only if $f(z_i) \approx f(z_j)$. Hence f acts as an automorphism on each fibre of $\psi^{-1}(U_i)$ and induces an automorphism on the fibres of B . Q. E. D.

Case I.

Assume that the group invariant G of the family is not trivial. This means that $e_2(B) = 12(p_g - q + 1) = 12(p_a + 1) > 0$, or equivalently, that there are singular fibres.

6.2. Lemma:

If an elliptic surface S has singular fibres, then

$$\dim \text{Aut}(S) = 0.$$

Proof:

If the dimension is greater than zero, then the dimension of the Lie algebra of holomorphic vector fields is positive. Since the Euler number is also positive, every vector field has zeros. But a theorem of Carrel, Howard and Kosniowski [5] shows that if a compact complex surface has a holomorphic vector field with zeros, then the surface is either of Kodaira class VII (i.e. $b_1(S)=1$), or a ruled surface or a rational surface. But if any of these is elliptic, we know that it has no singular fibres, or $c_2(S)=0$. Hence $\text{Aut}(S)$ is a discrete group. Q.E.D.

Next we want to see for what cases an automorphism will preserve the elliptic fibration. This is already true if the surface is not algebraic, since in that case, the elliptic fibration is unique as shown by Kodaira ([12] Thm. 4.3).

Suppose now that the canonical bundles of S and of B are denoted by K_S and K_B respectively. Then we know that there are line bundles on Δ denoted by K and L , where K is its canonical bundle and L is the normal bundle of $\sigma(\Delta)$ in B such that $K_S = \pi^*(K-L)$ and $K_B = \psi^*(K-L)$ ([13] Thm. 12 and [12] Thm. 12.1 resp.). Let $K-L = \sum n_z z$ with $z \in \Delta$, $n_z \in \mathbb{Z}$. Then the canonical divisors of B and S are given by

$$K_B = \sum n_z C_z = \psi^*(\sum n_z z), \quad K_S = \pi^*(\sum n_z z).$$

Since $c_1(K) = 2p-2$ and $c_1(L) = -(p_a+1)$ where p is the genus of Δ and p_a is the arithmetic genus of S or B (they agree), we have

$$\sum n_z = 2p - 2 + p_a + 1$$

By Jacobi's inversion theorem, if $\sum n_z > p$ (or $\sum n_z < -p$), then the line bundle $K-L$ comes

from an effective divisor (or from minus an effective divisor). This means that we can choose points $z \in \Delta$ such that $n_z > 0$ (or $n_z < 0$). This occurs if:

(*) $2p - 2 + p_s + 1 > p$ which happens if $p \geq 2$, or if $p=1$ and $p_s > 0$, or if $p=0$ and $p_s > 1$

(or (**) $2p - 2 + p_s + 1 < -p$ which happens if and only if $p=0$ and $p_s < 1$).

We conclude that the two exceptions to $\sum n_z > p$ (or $\sum n_z < -p$) are

i) $p=1$ and $p_s=0$,

and

ii) $p=0$ and $p_s=1$.

Assume now that S is an elliptic surface not satisfying i) and ii), and $f : S \rightarrow S$ is an automorphism. Then we claim that f maps fibres into fibres. If not, then since $C_z = \pi^{-1}(z)$ is a divisor of S , we have that $f(C_z)C_z > 0$ (i.e. the intersection of a fibre with the image f of a fibre under f is positive), and then $f(C_z)C_w > 0$ because any two fibres are homologous. Consider now the canonical bundle: $K_S = \sum n_z C_z$ with all $n_z \geq 0$ (or all $n_z < 0$ respectively). Then

$$0 = K_S K_S = K_S f(K_S) = (\sum n_z C_z)(\sum n_w f(C_w)) = \sum n_z n_w C_z f(C_w)$$

will be positive, giving a contradiction. Hence f maps fibres to fibres.

We consider now case i), so that Δ is an elliptic curve, and the arithmetic genus of S is 0. Recall that we are assuming the existence of at least one singular fibre. Then if $f : S \rightarrow S$ does not send fibres into fibres, we have that for some fibre C_z we have $C_z f(C_z) = n > 0$. Since the map $\pi f : C_z \rightarrow \Delta$ is not constant, so in fact it is onto. (Notice that for intersection properties, all fibres C_z and C_w are homologous, so that $f(C_z)C_w = f(C_z)C_z$). In particular, this is true for singular fibres too, but non-multiple singular fibres consist of the union of projective lines, and we can not map a projective line onto an elliptic curve. So necessarily f maps fibres into fibres, preserving the elliptic fibration of S .

Consider now case ii). Then Δ is a projective line, and the arithmetic genus of S is 1. Then the canonical bundle of S is the pull-back of the trivial line bundle of \mathbb{P}^1 . This

implies that $q(S)=0$, $p_g(S)=1$ and $c_2(S)=24$. Then S is a K3 surface.

Then we conclude:

6.3. Theorem:

If S is any elliptic surface without multiple singular fibres, $\pi:S \rightarrow \Delta$ the elliptic fibration, and B the basic member of the family containing S , $c_2(S)>0$ and S is not an algebraic K3 surface, then we have an exact sequence

$$0 \rightarrow H^0(\Delta, \Omega(B^\#)) \rightarrow \text{Aut}(S) \rightarrow G \rightarrow 0,$$

where G is a subgroup of the group of automorphisms of Δ .

6.4. Remarks:

If the elliptic K3 surface is non algebraic, we know that the elliptic fibration is unique. But there are examples of algebraic elliptic K3 surfaces with more than one elliptic surface structure. (c.f. Shioda and Inose [16]).

It may happen that the group $\text{Aut}(S)$ may be infinite discrete. (c.f. Shioda [15]).

Case II.

Suppose that S is an elliptic surface with no singular fibre, i.e. $c_2(S)=0$. Then S is a fibre bundle over Δ , where $\pi:S \rightarrow \Delta$ is the elliptic fibration with fibre C an elliptic curve. In this case the basic member of the family is $B=\Delta \times C$, $B^\#=B$. And we assume that $S=B^\sigma$ for some $\sigma \in H^1(\Delta, \Omega(B))$.

We want to find the surfaces of this class for which all automorphisms preserve the fibration. A similar argument as before will work. The canonical bundle is $K_S = \sum n_i C_i = \psi^*(K-L)$. Now, L is a topologically trivial line bundle. So if f is an automorphism of S , we have

$$0 = K_S^2 = K_S f(K_S) = \sum n_i n_w C_i f(C_w).$$

Now, the bundle $K-L$ comes from an effective divisor provided that $\deg(K-L) = \deg(K) \neq 0$ i.e. if Δ is not a complex torus. So in this case the automorphism $f: S \rightarrow S$ maps fibres into fibres and the coefficients n_i are all positive (or all negative), giving that $C_i f(C_i) = 0$. Hence f maps fibres to fibres preserving the elliptic fibration. Thus we have an exact sequence as in the last theorem.

6.5. Theorem:

Assume that S is an elliptic surface with no singular fibres, such that $S = B^c$, $B = \Delta \times C$ where Δ is a Riemann surface and C is analytically equivalent to a fibre. If S is not an abelian surface then the exact sequence holds:

$$0 \rightarrow H^0(\Delta, \Omega(B)) \rightarrow \text{Aut}(S) \rightarrow G \rightarrow 0,$$

where G is a subgroup of the group of automorphisms of Δ .

Notice that now the group $H^0(\Delta, \Omega(B))$ contains $\text{Aut}_0(C)$, and is of dimension at least one.

6.6. Remarks:

The exceptions for elliptic tori and elliptic K3 surfaces are real exceptions, as we can easily check by the following examples:

Let C be an elliptic curve. If $T = C \times C$ then $f(z, w) = (w, z)$ gives an automorphism that does not preserve the elliptic fibration.

For a K3 surface, consider the Kummer surface S obtained from T (by taking the desingularization of the quotient of $T/\langle j \rangle$ where $j(z, w) = -(z, w)$). Now the same map f as before commutes with the involution j , so f is well defined on S , and certainly it will not preserve the elliptic fibration on S induced from that on T .

Similar examples can be produced by considering $T = C \times H$, with C and H two elliptic curves with the property that there exists a map from C onto H .

7. Elliptic surfaces with multiple singular fibres

Let S be a relatively minimal elliptic surface with multiple singular fibres, and $\pi: S \rightarrow \Delta$ be the elliptic fibration. Suppose that the fibres $\{C_{a_\rho}\}$, $\rho = 1, 2, 3, \dots$ are the singular fibres and C_{a_ρ} are of type $m_\rho I_{b_\rho}$ for $1 \leq \rho \leq k$ where $m_\rho \geq 2$, and C_{a_ρ} are the simple fibres for $\rho > k$. Let m_0 be the least common multiple of m_1, m_2, \dots, m_k , $d = m_1 m_2 \dots m_k$, and choose arbitrarily $a_0 \in \Delta - \bigcup \{a_\rho\}$ to construct a d -fold abelian covering $\tilde{\Delta} \rightarrow \Delta$ which is unramified over $\Delta - \{a_0, a_1, \dots, a_k\}$, and has d/m_{a_ρ} branch points of order $m_\rho - 1$ over each a_ρ , for $\rho = 0, 1, \dots, k$. Let \tilde{S} be the analytic fibre space of elliptic curves over $\tilde{\Delta}$ induced from S , and denote by $\tilde{\pi}$ the canonical projection. \tilde{S} is an abelian cover of S which is unramified over $V - C_{a_0}$ and has d/m_0 branch curves over C_{a_0} . Let G be the covering transformation group of $\tilde{\Delta}$ with respect to Δ ; then G acts on \tilde{S} as a group of automorphisms and $S = \tilde{S}/G$. \tilde{S} is free from multiple fibres and if $\tilde{S} = B^\sigma$, it is known that $(S, G) = (B, G)^\sigma$, i.e. S is obtained from B^σ/G ; and G also acts as a group of automorphisms on B that leave invariant the curve $\sigma(\tilde{\Delta})$ contained in B . ([12] sect. 15)

7.1. Theorem:

Let S be an elliptic surface with multiple singular fibres. Assume that S is obtained from $(B, G)^\sigma$ as above. Except in the following two cases: S is algebraic and either $c_2(S) \leq 24$ and Δ is a projective line, or $c_2(S) = 0$ and Δ is an elliptic curve; we have an exact sequence

$$0 \rightarrow H^0(\tilde{\Delta}, \Omega(B^\#)) \rightarrow \text{Aut}(S) \rightarrow H \rightarrow 0,$$

where H is a subgroup of the group of automorphisms of Δ .

Proof:

We check first that any automorphism f of S maps fibres into fibres. For this, we follow essentially the same arguments as before. Consider the canonical bundle of S :

$$K_S = \psi^*(K_\Delta - L) + \sum (m_\rho - 1)P_\rho,$$

where P_ρ denotes the singular fibres with multiplicity m_ρ , K_Δ is the canonical bundle of

the base curve and L is a line bundle of Δ with $c_1(L) = -(p_s(S)+1)$. ([13] Thm. 12). We know that $\psi^*(K_\Delta - L) = \psi^*(\sum n_z z)$ for some $z \in \Delta$; and that the divisor $\sum n_z z$ is effective if the genus p of Δ is greater than 1. In this case, by considering intersection properties as before, f maps fibres into fibres.

If $p=1$ and S has simple singular fibres, they are projective curves and can not be mapped onto the base curve and then any automorphism maps fibres into fibres. Similarly if any of the P_i is projective. If this is not the case, then since $c_2(S)$ equals the sum of the Euler number of the singular fibres and they are all multiple elliptic curves, then it is zero and we can not conclude that fibres have to be mapped into fibres.

If the base curve is P^1 then the divisor $\sum n_z z$ is effective provided its degree $-2+p_s(S)+1$ is positive, i.e. $c_1(-L) = p_s + 1 > 2$. But $c_2(S) = 12(p_s + 1)$ (c.f. [12] formula 12.5 and [13] Thm.12). Thus the only possible exception for fibres to be mapped into fibres occurs if $c_2(S) \leq 24$.

Since the construction of $\tilde{S} \rightarrow \tilde{\Delta}$ depends canonically on $S \rightarrow \Delta$, any automorphism of S acting as the identity on the base curve lifts to an automorphism of \tilde{S} .

Finally we need to verify that the group of automorphisms that acts as automorphisms on each fibre is $H^0(\tilde{\Delta}, \Omega(B^\#))$. But notice that G acts on B leaving invariant the section $\sigma(\tilde{\Delta})$, that is, for any automorphism f induced by a section of $H^0(\tilde{\Delta}, \Omega(B^\#))$ and any $g \in G$, we have $fg = gf$. Hence f acts as an automorphism on $B^\sigma/G = S$ leaving the fibres fixed. And conversely, any automorphism f of S leaving the fibres fixed, lifts to an automorphism of B^σ , and since G does not move $\sigma(\tilde{\Delta})$ we have again that $fg = gf$ for any $g \in G$. Q. E. D.

8. Inoue surfaces

Inoue constructed some examples of surfaces of type VII₀ (i.e. $b_1=1$ and $p_g=0$) in [9]. An Inoue surface S is characterized by: i) $b_1(S)=1$ and $b_2(S)=0$, ii) S contains no curve, and iii) There exists a line bundle F of S such that $H^0(S, \Omega^1(F)) \neq 0$. In particular they are all non-Kaehler.

There are three types of these surfaces, to be described below. We follow Inoue [9] for the descriptions and notation.

8.1. Surfaces S_M

Let $M=(m_{ij}) \in SL(3, \mathbb{Z})$ be a unimodular matrix with eigenvalues $\alpha, \beta, \bar{\beta}$ such that $\alpha > 1, \beta \neq \bar{\beta}$. Choose a real eigenvector (a_1, a_2, a_3) and an eigenvector (b_1, b_2, b_3) of M corresponding to α and β respectively. Denote by \mathbb{H} the upper half-plane in \mathbb{C} , and let G_M be the group of holomorphic transformations of $\mathbb{H} \times \mathbb{C}$ generated by:

$$g_0(z, w) = (\alpha z, \beta w)$$

$$g_i(z, w) = (z + a_i, w + b_i), \quad i=1,2,3.$$

Then $S_M = \mathbb{H} \times \mathbb{C} / G_M$.

Inoue proves that $H^0(S_M, \Theta) = 0$, where Θ denotes the sheaf of holomorphic vector fields on S_M ([9] Prop. 2). This means that $\dim \text{Aut}(S_M) = 0$.

G has the following relations:

$$g_i g_j = g_j g_i \quad \text{for } i, j = 1, 2, 3.$$

$$g_0 g_i g_0^{-1} = g_1^{m_{i1}} g_2^{m_{i2}} g_3^{m_{i3}} \quad \text{for } i = 1, 2, 3.$$

We denote by Γ the subgroup generated by g_i for $i=1,2,3$.

The subgroups $\langle g_0 \rangle$ and Γ are normal in G . If $f \in \text{Aut}(S_M)$, then f lifts to an automorphism of $\mathbb{H} \times \mathbb{C}$, $\mathbb{H} \times \mathbb{C} / \langle g_0 \rangle$ and $\mathbb{H} \times \mathbb{C} / \Gamma$; since they are normal coverings of S_M . We denote by f any of these lifts, and denote by (z, w) the coordinates of $\mathbb{H} \times \mathbb{C}$. Then $f(z, w) = (f_1(z, w), f_2(z, w))$.

Notice that $f_1(z, w)$ does not depend of w , otherwise it would induce non-constant maps from \mathbb{H} onto \mathbb{C} which is impossible.

Also for each $z \in \mathbf{H}$ fixed, $f_2(z, _): \mathbf{C} \rightarrow \mathbf{C}$ is one-to-one and onto, so we have:
 $f_1(z, w) = az + b / bz + d$ with $a, b, c, d \in \mathbf{R}$ and $f_2(z, w) = \sigma(z)w + \tau(z)$ with $\sigma, \tau: \mathbf{H} \rightarrow \mathbf{C}$
 analytic. Then f must satisfy: $f g_0 = g_0^k f$, and $f g_i = g_i f$ for $i=1, 2$, or 3 , k a non-zero
 integer and $g \in \Gamma$. And any $g \in \Gamma$ is of the form $g(z, w) = (z + \sum n_i a_i, w + \sum n_i b_i)$.

Now

$$\begin{aligned} f g_0(z, w) &= f(\alpha z, \beta w) = ((a \alpha z + b) / (c \alpha z + d), \sigma(\alpha z) \beta w + \tau(\alpha z)) \\ &= g_0^r f(z, w) = (\alpha^r (az + b) / (cz + d), \beta^r (\sigma(z)w + \tau(z))) \end{aligned}$$

holds for all $(z, w) \in \mathbf{H} \times \mathbf{C}$. Then we obtain: $b = c = 0$, $\sigma(z)$ is constant, $r = 1$ and
 $\tau(z) = 0$.

Then $f(z, w) = (az, bw)$ for some $a \in \mathbf{R}$, $a > 0$ and $b \in \mathbf{C}$. But now,

$$\begin{aligned} f g_i(z, w) &= f(z + a_i, w + b_i) = (a(z + a_i), b(w + b_i)) = \\ &= g_i^{n_i} f(z, w) = (az + \sum n_i a_i, bw + \sum n_i b_i). \end{aligned}$$

Consequently $a = b = n_i$, implying that $f(z, w) = (nz, nw)$ with n a positive integer. Since f
 is invertible, we conclude that $n = 1$.

8.1.1. Theorem

If S_M is an Inoue surface as described above, then $\text{Aut}(S_M) = \{Id\}$.

8.2. Surfaces $S_{N, p, q, r, t}^{(+)}$

Let $N = (n_{ij}) \in SL(2, \mathbf{Z})$ be a unimodular matrix with two real eigenvalues α and $1/\alpha$
 with $\alpha > 1$. Choose real eigenvectors (a_1, a_2) , (b_1, b_2) of N corresponding to α and $1/\alpha$, and
 fix integers p, q, r ($r \neq 0$) and a complex number t . Let (c_1, c_2) be a solution of

$$(c_1, c_2) = (c_1, c_2) N^t + \frac{1}{r} (b_1 a_2 - b_2 a_1) (p, q),$$

where N^t denotes the transpose of N and

$$c_i = \frac{1}{2} n_{i1} (n_{i1} - 1) a_1 b_1 + \frac{1}{2} n_{i2} (n_{i2} - 1) a_2 b_2 + n_{i1} n_{i2} b_1 a_2 \quad (i = 1, 2).$$

Let $G = G_{N,p,q,r;t}^{(+)}$ be the group of holomorphic transformations of $\mathbb{H} \times \mathbb{C}$ generated by

$$\begin{aligned} g_0(z, w) &= (\alpha z, w + t) \\ g_i(z, w) &= (z + a_i, w + b_i z + c_i), \text{ for } i=1, 2. \\ g_3(z, w) &= (z, w + \frac{1}{r}(b_1 a_2 - b_2 a_1)). \end{aligned}$$

And define $S = S_{N,p,q,r;t}^{(+)} = \mathbb{H} \times \mathbb{C} / G$.

We have the following relations between the generators of G :

$$\begin{aligned} g_3 g_i &= g_i g_3 \text{ for } i=0, 1, 2, & g_1^{-1} g_2^{-2} g_1 g_2 &= g_3^r. \\ g_0 g_1 g_0^{-1} &= g_1^{n_{11}} g_2^{n_{12}} g_3^{n_{13}} & g_0 g_2 g_0^{-1} &= g_1^{n_{21}} g_2^{n_{22}} g_3^{n_{23}}. \end{aligned}$$

We consider as before the normal subgroups $\langle g_0 \rangle$ and Γ generated by g_i with $i=1, 2$ and 3.

As before we have for any automorphism f of S , $f g_0 = g_0^r$ and $f g_i = g f$ for some integer r , and some $g \in \Gamma$, and any element g of Γ is of the form $g(z, w) = (z + \sum n_i a_i, w + \sum m_j b_j z + K)$ with K some constant. And $f(z, w) = (az + b, \sigma(z)w + \tau(z))$.

As with S_M we get that $r=1$, $a=1$, and $b=0$; since the action on the first coordinates is of the same type. And also:

$$f g_0(z, w) = (\alpha z, \sigma(\alpha z)(w + t) + \tau(\alpha z)) = g_0 f(z, w) = (\alpha z, \sigma(z)w + \tau(z) + t).$$

That is

$$\sigma(z)w + \tau(z) + t = \sigma(\alpha z)(w + t) + \tau(\alpha z)$$

holds for all $(z, w) \in \mathbb{H} \times \mathbb{C}$. Thus $\tau \in \mathbb{C}$ is constant and $\sigma=1$.

Now we obtain that f actually commutes with the generators of G , so it induces an automorphism of S . I remark that Inoue proves (Prop. 3 in [9]) that $\dim H^0(S_{N,p,q,r;t}^{(+)}, \Theta) = 1$. We have proved:

8.2.1. Theorem

If $S = S_{N,p,q,r;t}^{(+)}$ is the Inoue surface described above then

$$\text{Aut}(S) = \{f(z, w) = (z, w + \tau) : \tau \in \mathbb{C}\}.$$

8.3. Surfaces $S_{N, p, q, r}^{(-)}$

Let $N = (n_{ij}) \in GL(2, \mathbb{Z})$ be a matrix with determinant -1 having real eigenvalues α , $-1/\alpha$ such that $\alpha > 1$. We choose real eigenvectors (a_1, a_2) , (b_1, b_2) of N corresponding to α and $-1/\alpha$ respectively, and fix integers p, q, r ($r \neq 0$). Define (c_1, c_2) to be the solution of

$$-(c_1, c_2) = (c_1, c_2)N^t + \frac{1}{r}(b_1 a_2 - b_2 a_1)(p, q)$$

where N^t denotes the transpose of N and

$$c_i = \frac{1}{2}n_{i1}(n_{i2}-1)a_1 b_1 + \frac{1}{2}n_{i2}(n_{i2}-1)a_2 b_2 + n_{i1}n_{i2}b_1 a_2 \quad (i=1,2).$$

Let $G = G_{N, p, q, r}^{(-)}$ denote the group of holomorphic transformations of $\mathbb{H} \times \mathbb{C}$ generated by

$$\begin{aligned} g_0(z, w) &= (\alpha z, -w), \\ g_i(z, w) &= (z + a_i, w + b_i + c_i) \quad \text{for } i=1,2; \\ g_3(z, w) &= (z, w + \frac{1}{r}(b_1 a_2 - b_2 a_1)). \end{aligned}$$

And define $S = S_{N, p, q, r}^{(-)} = \mathbb{H} \times \mathbb{C} / G$.

Then $S_{N, p, q, r}^{(-)}$ has an $S_{N, p, q, r, 0}^{(+)}$ as an unramified double covering surface, and $\dim H^0(S_{N, p, q, r}^{(-)}, \Theta) = 0$. (Prop. 5 in [9]).

We have the following relations in G :

$$\begin{aligned} g_3 g_i &= g_i g_3 \quad \text{for } i=1,2. & g_1^{-1} g_2^{-1} g_1 g_2 &= g_3^r, \\ g_0 g_1 g_0^{-1} &= g_1^{n_{11}} g_2^{n_{12}} g_3^p, & g_0 g_2 g_0^{-1} &= g_1^{n_{21}} g_2^{n_{22}} g_3^q, & g_0 g_3 g_0^{-1} &= g_3^{-1}. \end{aligned}$$

So that the group $\langle g_0 \rangle$ and $\Gamma = \langle g_1, g_2, g_3 \rangle$ are normal in G , as in the previous case.

If f is an automorphism of $S_{N, p, q, r}^{(-)}$, then f is of the form $f(z, w) = (z, w + \tau)$ and f commutes with g_0 (same computation as for the last case). And

$$f g_0(z, w) = (\alpha z, -w + \tau) = g_0 f(z, w) = (\alpha z, -w - \tau),$$

for all $(z, w) \in \mathbb{H} \times \mathbb{C}$, hence $\tau = 0$.

8.3.1. Theorem

The Inoue surface $S_{N,p,q,r}^{(-)}$ constructed above has no non-trivial automorphism.

Bibliography

- [1] Andreotti, A., Sopra le superficie algebriche che posseggono trasformazioni birazionali in se. *Univ. Roma Ist. Naz. Alta Mat. Rend. Mat. Pura e Appl.* 9 (1950) 255-279.
- [2] Bandman, T. M., Surjective holomorphic mappings of projective manifolds. *Siberian Math. Journal* 22 No. 2 (1981) 204-210.
- [3] Barth, W., and Peters, C., Automorphisms of Enriques surfaces. *Inv. Math* 73 (1983) 383-411.
- [4] Beauville, A., *Compact Complex Surfaces*. London Math. Soc. Series. No. 68 (1983).
- [5] Carrel, J., Howard, A., and Kosniowski, C. Holomorphic vector fields on complex surfaces. *Math. Ann.* 204 (1973) 73-81.
- [6] Fujiki, A., On the automorphism groups of compact Kaehler manifolds. *Inv. Math.* 44 (1978) 225-258.
- [7] Griffiths, P. and Harris, J., *Algebraic Geometry*. Addison-Wesley. (1978).
- [8] Howard, A., and Sommese, A. J., On the orders of the automorphism groups of certain projective manifolds. In *Manifolds and Lie groups*. Birkhauser, Progress in Math. (1981).
- [9] Inoue, M., On surfaces of class VII₀. *Inv. Math.* 24 (1974) 269-310.
- [10] Kobayashi, S., *Transformation Groups in Differential Geometry*. *Ergebnisse der Math.* Vol. 70. Springer-Verlag (1972).
- [11] Kobayashi, S. and Ochiai, T., Meromorphic mappings onto compact complex spaces of general type. *Inv. Math.* 31 (1975) 7-16.
- [12] Kodaira, K., On compact complex analytic surfaces. I, II and III. *Ann. of Math.* 71 (1960) 111-152, 77 (1963) 563-626, and 78 (1963) 1-40.
- [13] Kodaira, K., On the structure of compact complex analytic surfaces. I, II, III and IV. *Amer. J. of Math.* 86 (1964) 751-798, 88 (1966) 682-721, 90 (1968) 55-83 and 90

(1968) 1048-1066.

[14] Nikulin, V., Finite automorphism groups of Kaehler K3 surfaces. *Trans. Moscow Math. Soc.* 2 (1980) 71-135.

[15] Shioda, T., On elliptic modular surfaces. *J. Math. Soc. Japan* 24 No. 1 (1972) 20-59.

[16] Shioda, T. and Inose, H., On singular K3 surfaces. In *Complex Analysis and Algebraic Geometry*, papers in honor of K. Kodaira (1977). Ed. by Baily and Shioda.